Continuum Physics

Problem set 6

Problem 1.

Demonstrate whether the following strain energy functions for solids satisfy the following requirements: (a) Frame invariance with respect to rigid body motions on the current configuration. (b) Material isotropy. In each case, derive the expression for the first Piola-Kirchhoff stress tensor: $\boldsymbol{P} = \partial W/\partial \boldsymbol{F}$, where $\boldsymbol{F}$ is the deformation gradient. In what follows, $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$ is the right Cauchy-Green tensor, $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the left Cauchy-Green tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{1})$ is the Green-Lagrange strain tensor, and I_A, II_A, III_A are the first, second and third principal invariants of the tensor, $\boldsymbol{A}$. The principal stretches are $\lambda_1, \lambda_2, \lambda_3$. Finally, all logarithms are with respect to base e.

(i) The St. Venant-Kirchhoff model: $W(\boldsymbol{E}) = \frac{1}{2}\boldsymbol{E}\colon\mathbb{C}\colon\boldsymbol{E}$, where the fourth-order elasticity tensor is $\mathbb{C} = \lambda\boldsymbol{1}\otimes\boldsymbol{1} + 2\mu\mathbb{I}$, and $\mathbb{I}_{IJKL} = \frac{1}{2}(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK})$. Note that λ here is the Lamé constant, not to be confused with a principal stretch.

(ii) The St. Venant-Kirchhoff model extended to cubic crystals: $W(\boldsymbol{E}) = \frac{1}{2}\boldsymbol{E}\colon\mathbb{C}\colon\boldsymbol{E}$, where the fourth-order elasticity tensor is $\mathbb{C} = \alpha(\boldsymbol{A}_1\otimes\boldsymbol{A}_1 + \boldsymbol{A}_2\otimes\boldsymbol{A}_2 + \boldsymbol{A}_3\otimes\boldsymbol{A}_3) + \beta(\boldsymbol{1}\otimes\boldsymbol{1} - \boldsymbol{A}_1\otimes\boldsymbol{A}_1 - \boldsymbol{A}_2\otimes\boldsymbol{A}_2 - \boldsymbol{A}_3\otimes\boldsymbol{A}_3) + 2\mu(\mathbb{I} - \boldsymbol{A}_1\otimes\boldsymbol{A}_1 - \boldsymbol{A}_2\otimes\boldsymbol{A}_2 - \boldsymbol{A}_3\otimes\boldsymbol{A}_3)$, $\mathbb{I}_{IJKL} = \frac{1}{2}(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK})$, $\boldsymbol{A}_1 = \boldsymbol{e}_1\otimes\boldsymbol{e}_1$, $\boldsymbol{A}_2 = \boldsymbol{e}_2\otimes\boldsymbol{e}_2$ and $\boldsymbol{A}_3 = \boldsymbol{e}_3\otimes\boldsymbol{e}_3$. Here, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are the constant, orthonormal basis vectors. The constants α, β and μ bear no relation to each other.

(iii) The quadratic-logarithmic model, used for metals: $W(\lambda_1,\lambda_2,\lambda_3) = \frac{1}{2}\lambda(\log\lambda_1 + \log\lambda_2 + \log\lambda_3)^2 + \mu[(\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2]$.

(iv) The compressible neo-Hookean model, used for polymers: $W(\boldsymbol{C}) = \frac{1}{4}\lambda(III_C - 1) - \frac{1}{2}(\frac{1}{2}\lambda + \mu)(\log III_C) + \frac{1}{2}\mu(I_C - 3)$. Again, λ is the Lamé constant, not to be confused with a principal stretch.

(v) A model used for soft biological tissue, encompassing the "entropic elasticity" response of long-chain molecules:

$$
\begin{aligned}
W(\boldsymbol{F}) \quad = \quad & \underbrace{\frac{Nk_B\theta}{4A}\left(\frac{2r^2}{L} + \frac{L}{1 - r/L} - r\right)}_{\text{chain molecule term}} \\
& \underbrace{-\frac{Nk_B\theta}{4A}\left(\frac{1}{L} + \frac{1}{4r_0(1-\frac{r_0}{L})^2} - \frac{1}{4r_0}\right)4r_0^2\log\frac{r}{r_0}}_{\text{repulsive term}} \\
& \underbrace{+\frac{\gamma}{\beta}(J^{-2\beta} - 1) + 2\gamma\boldsymbol{1}\colon\frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{1})}_{\text{bulk compressibility}}.
\end{aligned}
$$

Here, r and r_0 are the final and initial lengths of the long chain molecule, and are related by $r = r_0\sqrt{\boldsymbol{M}\cdot\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{M}}$, with $\boldsymbol{M}$ being the orientation of the long chain molecule in the reference configuration (say, before deformation). Additionally, the following constant scalars appear in the model: N is the number of long chain molecules per unit volume, k_B is the Boltzmann constant, θ is the temperature, A is the persistence length of the molecules, L is the contour length of the molecules, γ and β are constants.

(vi) One I cooked up just now: $W(\boldsymbol{b}) = \alpha \boldsymbol{A} \colon \boldsymbol{b}$, where $\boldsymbol{A}$ is some constant second-order tensor, and α is a scalar constant.