Continuum Physics

Midterm Exam

There are 3 questions, one one each page. All questions carry equal weight. Your answer should appear immediately below each question. Use the back of Page i to continue your answer to Question i, for $i = 1, 2, 3$. Use your own extra sheets if needed.

Problem 1.

$\boldsymbol{A} \in \mathbb{GL}(3)$ is a real, symmetric tensor. Its principal invariants are $I_1 = \mathrm{trace}(\boldsymbol{A})$, $I_2 = \frac{1}{2}((\mathrm{trace}(\boldsymbol{A}))^2 - \mathrm{trace}(\boldsymbol{A})^2)$ and $I_3 = \det(\boldsymbol{A})$. Its eigen values, λ, satisfy the characteristic equation:

$$\lambda^3 - \lambda^2 I_1 + \lambda I_2 - I_3 = 0 \tag{1}$$

Show that $\boldsymbol{A}$ itself satisfies the following tensorial equation: equation:

$$\boldsymbol{A}^3 - \boldsymbol{A}^2 I_1 + \boldsymbol{A} I_2 - I_3 \mathbf{1} = \mathbf{0}, \tag{2}$$

where $\mathbf{1}$ is the second-order isotropic tensor.

This is the Cayley-Hamilton Theorem restricted to real, symmetric tensors.

Problem 2.

Consider the following motion:

$$\boldsymbol{\varphi}(\boldsymbol{X}, t) = (\alpha \cdot t)\boldsymbol{Q}(t)\boldsymbol{X} \tag{3}$$

where α is a constant scalar and $\boldsymbol{Q}(t) \in \mathrm{SO}(3)$.

(a) Find the spatial velocity $\boldsymbol{v}(\boldsymbol{x}, t)$.

(b) Find the spatial acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$. What is the physical significance of the various contributions to $\boldsymbol{a}(\boldsymbol{x}, t)$?

Problem 3.

Consider a body, which in the reference configuration, Ω_0, is a sphere of radius R. Its deformation gradient is $\boldsymbol{F}$.

(a) If $\boldsymbol{F} = \lambda \mathbf{1}$, where $\mathbf{1}$ is the usual second-order isotropic tensor, what is the shape of the body in its deformed configuration, Ω_t?

(b) If $\boldsymbol{F} = \lambda_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$, where $\lambda_1 \neq \lambda_2 \neq \lambda_3$ are constants and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a constant orthonormal basis, then what is the shape of the deformed configuration, Ω_t? Provide an explicit parametrization for Ω_t.